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Output Tracking of Uncertain Fractional-order Systems via Robust Iterative Learning Sliding Mode Control

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Abstract – This paper develops a novel controller called iterative learning sliding mode (ILSM) to control linear and nonlinear fractional-order systems. This control applies a combination structures of continuous and discontinuous controller, conducts the system output to the desired output and achieve better control performance. This controller is designed in the way to be robust against the external disturbance. It also estimates unknown parameters of fractional-order systems. The proposed controller unlike the conventional iterative learning control for fractional systems does not need to apply direct control input to output of the system. It is shown that the controller perform well in partial and complete observable conditions. Simulation results demonstrate very good performance of the iterative learning sliding mode controller for achieving the desired control objective by increasing the number of iterations in the control loop.

Keywords: Fractional order systems, Sliding mode control, Iterative learning technique, Robust control, Partial and complete observability

1. Introduction

Iterative Learning Control (ILC) is one of the recent topics in control theories. ILC, which belongs to the intelligent control methodology, is an approach to improve the transient performance of systems that operate repetitively over a fixed time interval. In detail, they apply a fixed-length input signal to a certain system. After the complete input is applied, the system returns to the same initial state and the output trajectory that resulted from the applied input is compared with the desired reference. The possible error is used to construct a new input signal of the same length that is applied to the next iteration. The aim of the ILC algorithm is to continue the trial so that as more trials are executed, the output would approach the desired trajectory more [1].

Recently, an advanced calculus called fractional order calculus is applied in many controllers [2-6], as well as iterative learning control to improve their performance. Fractional calculus is an old mathematical operation with a 300-year-old history [7]. For many years, this branch of science has been considered as a pure mathematical and theoretical discipline with nearly no application [8]. It also has been found that the behavior of many physical systems can be properly described by using the fractional-order system theory [9-11]. In fact, most of real world processes can better modeled by fractional-order systems [12] and the fractional controller has shown better performance for such system [13-20]. In [1], the authors have defined-type ILC algorithm for linear fractional-order systems. In this paper the designed controller is not robust and can be implemented only for linear systems. The system dynamic is also without disturbance and the parameters of the system assumed to be fully known. Process can only track the desired output if the control signal is directly applied to the output.

PD^{α} -type and P-type ILC algorithms are designed for nonlinear fractional-order systems in [21] and [22] respectively. In both papers the designed controllers are not robust, system dynamics have no unknown parameters and can be implemented only for special class of nonlinear systems. For the first time, the robust controller with ILC structure for fractional system with D^{α} -type and adaptive

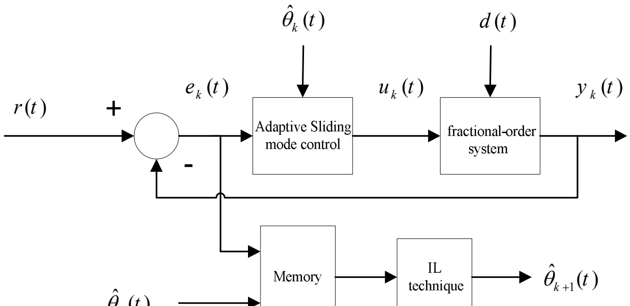

Fig. 1. The basic scheme of ILSM control

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P-type based have been introduced in [23] and [24], respectively. The introduced methods have however some restrictions. Firstly, for a desired output tracking process the controller input must be fed to output dynamic. Secondly, system states dynamic is assumed without any unknown parameters.

The basic idea of ILSM control is illustrated in Fig. 1, where u_k and y_k are, respectively, the system input and output in the k_{th} iteration, $\hat{\theta}_k(t)$ and $\hat{\theta}_{k+1}(t)$ are the recursive control part of the k_{th} and $(k+1)_{th}$ trial, that are used to learn the unknown parameters and $r(t)$ is the given desired output. The goal of ILSM is that $\lim_{k\to\infty} y_k(t) = r(t)$ for all $t \in [0,T]$, where T is a fixed constant.

The advantages of using sliding mode controller is being robust in presence of disturbance and also having similar structure for linear and nonlinear systems. So far, ILC has been applied in PID to control fractional-order systems [1, 21-24], in which convergence is a key component of the design. Furthermore this condition is different in linear and nonlinear systems which makes use of such structure impossible to control both linear and nonlinear system simultaneously and also is not robust in presence of unknown disturbance. These are the limitations that will be overcome with our proposed method.

The mentioned controller in this paper, guarantees the convergence of tracking error using iterative learning algorithm in presence of unknown parameters at system dynamic. In addition, model reference adaptive control (MRAC) is applied in the sliding mode control structure. By that, any disturbance will be rejected and system will be kept at the desired sliding surface without having any information on disturbance dynamic. In addition to the mentioned advantages, the proposed control unlike the conventional ILC based control such as PIDILC [1, 21-24] does not require be applied directly to the output of the system while guaranteeing the desired performance.

This paper is organized as follows: in Section 2, basic definitions of fractional calculus is presented. System description and its assumptions are expressed in Section 3. ILSM scheme as well as the convergent condition for fractional-order systems are discussed in Section 4. MATLAB/SIMULINK results for two benchmarks (Duffing and Chaotic oscillator systems) are shown in Section 5. And finally, some conclusions are drawn in Section 6.

2. Preliminaries

There are several definitions for fractional calculus, but two of them are more popular, which are defined as follows:

Definition 1 [7]: The Caputo fractional derivative and integral of order α of function $f(t)$ at a time instant $t \geq 0$ are defined as:

$$
{}_cD_x^{\alpha} f(x) = \begin{cases} \left(\dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_c^x (x-\xi)^{-\alpha-1} f(\xi) d\xi \;,\; if \;\; \alpha < 0 \right) \\ f(x), \;\; if \;\; \alpha = 0 \\ \left({}_cD_x^{\alpha-n}\left[D^n f(x) \right] \;,\; n = \min\{k \in N : k > \alpha\} \;,\; if \, \alpha > 0 \right) \end{cases} \tag{1}
$$

where $\Gamma(.)$ is a well-known function, called Euler's gamma function:

$$
\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt \tag{2}
$$

Definition 2 [7]: The Riemann-Liouville fractional derivative and integral of order of function $f(t)$ at a time instant $t \geq 0$ defined as:

$$
{}_cD_x^{\alpha} f(x) = \begin{cases} \left(\dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_c^x (x-\xi)^{-\alpha-1} f(\xi) d\xi \;,\; if \;\; \alpha < 0 \right) \\ f(x), \;\; if \;\; \alpha = 0 \\ \left(D^n\left[{}_cD_x^{\alpha-n} f(x) \right] \;,\; n = \min\{k \in N : k > \alpha\} \;,\; if \, \alpha > 0 \right) \end{cases} \tag{3}
$$

In the rest of this paper, the notation $D^{\alpha}(.)$ indicates the Riemann-Liouville derivative of order α .

Definition 3: Complete observability is defined that all variable states exist in output dynamic and their variables directly affect the output.

Definition 4: Partial observability is defined that some of the variable states exist in output dynamic and the variables of the other states that are not in output dynamic

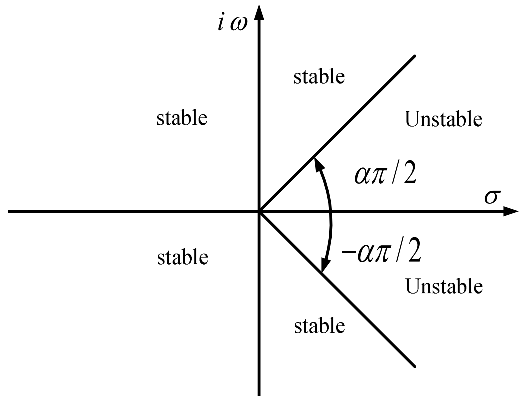

Fig. 2. Stability domain for FO linear systems with $0 < \alpha < 1$

do not directly affect the output.

Lemma [25]: Consider the following linear fractional order autonomous system:

$$D^{\alpha}x(t) = Ax(t) \quad x(0) = x_0 \tag{4}$$

where $0 < \alpha < 1$, $x(t) \in R^n$ are states vector and $A \in R^n$ is a constant matrix. This system is asymptotically stable if $|\arg(\lambda(A))| > \alpha\frac{\pi}{2}$, according Fig. 2. λ represents the eigenvalues of matrix *A*.

3. System description

Consider the following higher-order single-input and single-output linear/nonlinear fractional-order dynamic system described by:

$$\begin{aligned} &D^{\alpha}x_i(t) = x_{i+1}(t) \\ &D^{\alpha}x_n(t) = F(x,t) + \theta^T\xi(x) + b(x,t)u(t) + d(t) \\ &y(t) = \sum_{i=1}^{q} c_i x_i(t) \end{aligned} \tag{5}$$

where the measurable system states $x(t) = [x_1, x_2, \ldots x_n]^T$, $u(t)$ is a control input, $y(t)$ is a system output, $F(x,t)$ is a linear/nonlinear certain dynamic system, θ is an unknown and constant vector with $l\times 1$ dimension to be learnt, $\xi(x)$ is a function of state variables with $l\times 1$ dimension, $b(x,t)$ is a known non-zero function and the variable $d(t)$ represents the disturbance with unknown dynamics. c_i is a constant row vector and assumes that q is maximum value of i such that $c_i \neq 0$ (for $i = 0,1,\ldots,n$). So if $q = n$, the system is considered complete observable and if $q < n$, then the system is considered partial observable.

Assumption 1. The unknown disturbance variable is bounded such that:

$$|d(t)| \le M \quad \forall t \in [0,T]$$

where *M* is a known positive constant.

Assumption 2. The desired output trajectory $r(t)$ is bounded and differentiable with respect to time *t* up to the *n*th order on a finite time interval $[0,T]$, and all of the fractional derivatives are available and bounded.

From assumption 2 consider (6):

$$\begin{aligned} &D^{\alpha}r_i(t) = r_{i+1}(t) \qquad r_1(t) = r(t) \\ &D^{\alpha}y_i(t) = y_{i+1}(t) \qquad y_1(t) = y(t) \end{aligned} \tag{6}$$

Define fixed parameter $\eta = n - q + 1$ and for $\eta > 1 (n > q)$ using (5) and (6), the error vector is defined as follow:

$$\begin{aligned} &e_1(t) = r_1(t) - y_1(t) = r(t) - \sum_{i=1}^{q} c_i x_i(t) \\ &e_2(t) = r_2(t) - y_2(t) = r_2(t) - \sum_{i=1}^{q} c_i x_{i+1}(t) \\ &\vdots \\ &e_{\eta}(t) = r_{\eta}(t) - y_{\eta}(t) = r_{\eta}(t) - \sum_{i=1}^{q} c_i x_{i+\eta-1}(t) \end{aligned} \tag{7}$$

From (7) error equations can be taken as follow:

$$\begin{aligned} &D^{\alpha}e_1(t) = e_2(t) \\ &D^{\alpha}e_2(t) = e_3(t) \\ &\vdots \\ &D^{\alpha}e_{\eta}(t) = D^{\alpha}r_{\eta}(t) - D^{\alpha}\Big(\sum_{i=1}^{q} c_i x_{i+\eta-1}(t)\Big) \\ &\quad = D^{\alpha}r_{\eta}(t) - D^{\alpha}\Big(\sum_{i=1}^{q-1} c_i x_{i+\eta-1}(t)\Big) \\ &\quad - c_q\Big(F(x,t) + \theta^T\xi(x) + b(x,t)u(t) + d(t)\Big) \end{aligned} \tag{8}$$

Assumption 3. The initial condition $D^{\alpha-1}e_1(0) = D^{\alpha-1}e_2(0) = \cdots D^{\alpha-1}e_n(0) = 0$ at any iteration $\forall t \in [0,T]$, such that the sliding surface $S(0) = 0$.

4. Main Result

The main task in this paper designs a robust controller for system (5) such that the output $y(t)$ tracks a time variable reference signal $r(t)$ and error asymptotically tends to zero. For this purpose a new design of ILSM control is proposed. This new algorithm uses a combined time-domain and iteration-domain law allowing to guarantee the boundedness of the tracking error and the control input, as well as the convergence of the tracking error to zero, without any a priori knowledge of fractional system parameters. Also the mentioned controller is a robust in presence of external disturbance without knowing the details of disturbance dynamic.

4.1 ILSM controller design

For the considered system (5), an integral sliding surface dynamic is chosen as follow:

$$S(t) = k_1 D^{\alpha-1}e_1(t) + k_2 D^{\alpha-1}e_2(t) + \cdots k_{\eta} D^{\alpha-1}e_{\eta}(t) \tag{9}$$

Maintaining the system's states on this surface results as:

$$\begin{aligned} &S(t) = 0 \Rightarrow k_1 D^{\alpha-1}e_1(t) + k_2 D^{\alpha-1}e_2(t) + \cdots k_{\eta} D^{\alpha-1}e_{\eta}(t) = 0 \\ &\quad \Rightarrow k_{\eta} D^{\alpha-1}e_{\eta}(t) = -D^{\alpha-1}\sum_{i=1}^{\eta-1} k_i e_i(t) \end{aligned} \tag{10}$$

Due to the use of Riemann–Liouville definition, $D^{1-\alpha}$ is derived from the parties of the above equation:

$$k_\eta e_\eta = -\sum_{i=1}^{\eta-1} k_i e_i \tag{11}$$

From (8) the following relations can be obtained:

$$\begin{aligned} D^\alpha e_1(t) &= e_2(t) \\ D^\alpha e_2(t) &= e_3(t) \\ &\vdots \\ D^\alpha e_{\eta-1}(t) &= e_\eta(t) \end{aligned} \tag{12}$$

where $e_\eta(t)$ is viewed as a control input, the task is to design $e_\eta(t)$ to stabilize the origin (equilibrium point) of system (12). This task may be achieved by choosing:

$$\begin{aligned} &e_\eta(t) = -(k_1 e_1(t) + k_2 e_2(t) + \cdots k_{\eta-1} e_{\eta-1}(t)) \\ &\Rightarrow e_\eta(t) = -\sum_{i=1}^{\eta-1} k_i e_i(t) \end{aligned} \tag{13}$$

By substituting (13) in (12), the following formula will be given:

$$\begin{aligned} &D^\alpha e_{i-1}(t) = A e_i(t) \qquad (i = 2,3,\ldots,\eta) \\ &A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \ddots & & \\ -k_1 & -k_2 & -k_3 & \cdots & -k_{\eta-1} \end{bmatrix} \end{aligned} \tag{14}$$

where k_1 to $k_{\eta-1}$ are chosen such that the eigenvalues of the matrix A satisfy the stability condition of Lemma. In this situation, the linear fractional order error system (14), is asymptotically stable and the components of the error vector decay toward zero. So with $k_\eta = 1$ in (11), it's obvious that for $S(t) = 0$ the error vector is toward zero and therefore, output tracking of the time-varying reference signal is achieved.

Taking derivatives with respect to time *t* on both sides of (9) where $k_\eta = 1$, (15) is obtained:

$$\dot{S}(t) = k_1 D^\alpha e_1(t) + k_2 D^\alpha e_2(t) + \cdots + D^\alpha e_\eta(t) \tag{15}$$

Considering the fact that $e(t) = y(t) - r(t)$, the above equation can be further expanded:

$$\begin{aligned} \dot{S}(t) = &\sum_{i=1}^{\eta-1} k_i e_{i+1}(t) + D^\alpha r_\eta(t) - \sum_{i=1}^{q-1} c_i x_{i+\eta-1}(t) \\ &- c_q(F(x,t) + \theta^T \xi(x) + b(x,t)u(t) + d(t)) \end{aligned} \tag{16}$$

By taking $c_q = 1$, the above equation is as bellow:

$$\begin{aligned} \dot{S}(t) = &\sum_{i=1}^{\eta-1} k_i e_{i+1}(t) + D^\alpha r_\eta(t) - \sum_{i=1}^{q-1} c_i x_{i+\eta-1}(t) \\ &- F(x,t) - \theta^T \xi(x) - b(x,t)u(t) - d(t) \end{aligned} \tag{17}$$

The above equation can be interpreted as the sliding variable dynamics. The condition $S(t) = 0$ defines the system motion on the sliding surface. The control signal $u(t)$ is designed as an iterative and continuous control input signal. Therefore, an ILSM control at kth iteration is designed as follow:

$$u_k(t) = b^{-1}(x_k,t) \begin{pmatrix} \sum_{i=1}^{\eta-1} k_i e_{i+1,k}(t) + D^\alpha r_\eta(t) \\ -\sum_{i=1}^{q-1} c_i x_{i+\eta-1,k}(t) \\ -F(x_k,t) - \hat{\theta}_k^T \xi(x_k) \\ -M(\psi_k(t) - 1)\operatorname{sgn}(S_k(t)) \end{pmatrix} \tag{18}$$

where *k* indicates the number of iterations, sgn is the signum function, *M* is the upper bound of uncertainty that was said in assumption 1 and $\hat{\theta}(t)$ is the iterative learning control part that is used to learn the unknown parameter θ and generated by the following recursive law.

$$\hat{\theta}_k(t) = \hat{\theta}_{k-1}(t) - \xi(x_k)\beta S_k(t) \tag{19}$$

where β is defined as the learning gain.

The initial value of the parameter vector is set to $\hat{\theta}_k(0) = \hat{\theta}_{k-1}(T)\ \forall k = 1,2,\ldots$, and the initial parameter profile for *k*=0 is chosen as $\hat{\theta}_0(0) = \theta_0\ \forall t \in [0,T]$, where is a constant parameter vector. The reason for choosing this initial condition for parameter update mechanism in (19) is that a constant parameter will hold the same value at *t*=0 and *t*=*T*. If $\hat{\theta}_k(0) \neq \hat{\theta}_{k-1}(T)$, it would be meaningless to apply the consecutive initial condition. The consecutive initial condition is applicable to different types of updating mechanism, only if we have additional knowledge that $\hat{\theta}_k(0) = \hat{\theta}_{k-1}(T)$ [26].

The variable $\psi_k(t)$ is used to attenuate the effect of the unknown disturbance. This variable is defined as below:

$$\dot{\psi}_k(t) = -M\left|S_k(t)\right| \qquad \psi_k(0) = 0 \tag{20}$$

where, *M* is upper bound of uncertainty that was said in assumption 1.

Therefore, the sliding surface dynamics (17) can be simplified by inserting the ILSM law (18):

$$\begin{aligned} \dot{S}_k(t) = &\phi_k^T(t)\xi(x_k) \\ &+ M(\psi_k(t) - 1)\operatorname{sgn}(S_k(t)) - d(t) \end{aligned} \tag{21}$$

where $\phi_k(t) = \hat{\theta}_k(t) - \theta$ is the parametric estimation error.

4.2 Convergence of the output-tracking error

The following theorem constitutes the convergence of the sliding surface dynamics and output tracking error when the ILSM control is applied to the system.

Theorem 1: Consider the fractional order system (5) under the adaptive robust control torque (18), (20) and parameter recursive law (19). If assumptions (A1)-(A3) are satisfied, then the sliding surface will converge to a neighborhood of the origin.

Proof. To evaluate the convergence property cited in theorem 1, we define the following composite energy function at *k*th iteration for $t \in [0,T]$.

$$\begin{aligned} W_k(t) &= W_k^1(t)+W_k^2(t)+W_k^3(t) \\ &= \frac{1}{2}S_k^2(t)+\frac{1}{2}\psi_k^2(t)+\frac{1}{2\beta}\int_0^t \phi_k^T(\tau)\phi_k(\tau)d\tau \end{aligned} \tag{22}$$

The proof consists of two steps. Step1 derives the difference of composite energy and step 2 proves the convergence of the tracking error.

Step 1: *Derives the difference of composite energy*

Consider the difference of the first energy function between *k*th and (*k*-1)th iterations:

$$\begin{aligned} \Delta W_k^1(t) &= W_k^1(t)-W_{k-1}^1(t) \\ &= \frac{1}{2}S_k^2(t)-\frac{1}{2}S_{k-1}^2(t) \\ &= \int_0^t S_k(\tau)\dot{S}_k(\tau)d\tau-\frac{1}{2}S_{k-1}^2(t) \end{aligned} \tag{23}$$

Considering the fact that $S_k(t)\mathrm{sgn}(S_k(t)) = |S_k(t)|$ and by substituting the derivative of sliding surface proposed in (21) in to (23), it is obtained:

$$\begin{aligned} \Delta W_k^1(t) &= \int_0^t S_k(\tau)\phi_k^T(\tau)\xi(x_k(\tau))d\tau \\ &\quad +M\int_0^t \psi_k(\tau)|S_k(\tau)|d\tau - M\int_0^t |S_k(\tau)|d\tau \\ &\quad -\int_0^t d(\tau)S_k(\tau)d\tau-\frac{1}{2}S_{k-1}^2(t) \\ &\le \int_0^t S_k(\tau)\phi_k^T(\tau)\xi(x_k(\tau))d\tau \\ &\quad +M\int_0^t \psi_k(\tau)|S_k(\tau)|d\tau - M\int_0^t |S_k(\tau)|d\tau \\ &\quad +M\int_0^t |S_k(\tau)|d\tau-\frac{1}{2}S_{k-1}^2(t) \\ &\le \int_0^t S_k(\tau)\phi_k^T(\tau)\xi(x_k(\tau))d\tau \\ &\quad +M\int_0^t \psi_k(\tau)|S_k(\tau)|d\tau-\frac{1}{2}S_{k-1}^2(t) \end{aligned} \tag{24}$$

The difference of the second energy function between *k*h and (*k*-1)th iterations can be expressed as:

$$\begin{aligned} \Delta W_k^2(t) &= W_k^2(t)-W_{k-1}^2(t) \\ &= \frac{1}{2}\psi_k^2(t)-\frac{1}{2}\psi_{k-1}^2(t) \\ &= \int_0^t \psi_k(\tau)\dot{\psi}_k(\tau)d\tau-\frac{1}{2}\psi_{k-1}^2(t) \end{aligned} \tag{25}$$

Substituting (20) in to the (25), the above equation can be rearranged as:

$$\Delta W_k^2(t) = -M\int_0^t \psi_k(\tau)|S_k(\tau)|d\tau-\frac{1}{2}\psi_{k-1}^2(t) \tag{26}$$

At least, the difference of the third energy function between *k*th and (*k*-1)th iterations has the following form:

$$\begin{aligned} \Delta W_k^3(t) &= W_k^3(t)-W_{k-1}^3(t) \\ &= \frac{1}{2\beta}\int_0^t \phi_k^T(\tau)\phi_k(\tau)d\tau-\frac{1}{2\beta}\int_0^t \phi_{k-1}^T(\tau)\phi_{k-1}(\tau)d\tau \end{aligned} \tag{27}$$

The following equation is stablished:

$$\begin{aligned} &\frac{1}{2\beta}(\phi_k^T(t)\phi_k(t)-\phi_{k-1}^T(t)\phi_{k-1}(t)) \\ &= \frac{1}{2\beta}\phi_k^T(t)\phi_k(t)-\frac{1}{2\beta}\phi_{k-1}^T(t)\phi_{k-1}(t) \\ &= \frac{1}{\beta}\phi_k^T(t)\phi_k(t)-\frac{1}{\beta}\phi_k^T(t)\phi_{k-1}(t) \\ &\quad -\frac{1}{2\beta}\phi_k^T(t)\phi_k(t)-\frac{1}{2\beta}\phi_{k-1}^T(t)\phi_{k-1}(t)+\frac{1}{\beta}\phi_k^T(t)\phi_{k-1}(t) \end{aligned} \tag{28}$$

(28) can be calculated as follows:

$$\begin{aligned} &\frac{1}{2\beta}(\phi_k^T(t)\phi_k(t)-\phi_{k-1}^T(t)\phi_{k-1}(t)) \\ &= \frac{1}{\beta}\phi_k^T(t)(\phi_k(t)-\phi_{k-1}(t)) \\ &\quad -\frac{1}{2\beta}(\phi_k(t)-\phi_{k-1}(t))^T(\phi_k(t)-\phi_{k-1}(t)) \\ &= \frac{1}{\beta}\phi_k^T(t)(\hat{\theta}_k(t)-\hat{\theta}_{k-1}(t)) \\ &\quad -\frac{1}{2\beta}(\hat{\theta}_k(t)-\hat{\theta}_{k-1}(t))^T(\hat{\theta}_k(t)-\hat{\theta}_{k-1}(t)) \end{aligned} \tag{29}$$

Consider updating law (19), the above equation can be expanded as:

$$\begin{aligned} &\frac{1}{2\beta}(\phi_k^T(t)\phi_k(t)-\phi_{k-1}^T(t)\phi_{k-1}(t)) \\ &= -\frac{1}{\beta}\phi_k^T(t)(\xi(x_k(t))\beta S_k(t)) \\ &\quad -\frac{1}{2\beta}(\hat{\theta}_k(t)-\hat{\theta}_{k-1}(t))^T(\hat{\theta}_k(t)-\hat{\theta}_{k-1}(t)) \end{aligned} \tag{30}$$

Therefore, by substituting (30) in (27), we have:

$$\begin{aligned}\Delta W_k^3(t) = &-\int_0^t \phi_k^T(\tau)\xi(x_k(\tau))S_k(\tau)d\tau \\ &-\frac{1}{2\beta}\int_0^t (\hat{\theta}_k(\tau)-\hat{\theta}_{k-1}(\tau))^T(\hat{\theta}_k(\tau)-\hat{\theta}_{k-1}(\tau))d\tau\end{aligned} \tag{31}$$

The total energy function can be obtained by adding all of them:

$$\begin{aligned}&\Delta W_k(t) = \Delta W_k^1(t) + \Delta W_k^2(t) + \Delta W_k^3(t) \\ &\le \int_0^t S_k(\tau)\phi_k^T(\tau)\xi(x_k(\tau))d\tau \\ &+M\int_0^t \psi_k(\tau)\left|S_k(\tau)\right|d\tau - \frac{1}{2}S_{k-1}^2(t) \\ &-M\int_0^t \psi_k(\tau)\left|S_k(\tau)\right|d\tau - \frac{1}{2}\psi_{k-1}^2(t) \\ &-\int_0^t \phi_k^T(\tau)\xi(x_k(\tau))S_k(\tau)d\tau \\ &-\frac{1}{2\beta}\int_0^t (\hat{\theta}_k(\tau)-\hat{\theta}_{k-1}(\tau))^T(\hat{\theta}_k(\tau)-\hat{\theta}_{k-1}(\tau))d\tau\end{aligned} \tag{32}$$

(32) can be simplified as follows:

$$\begin{aligned}\Delta W_k(t) \le &-\frac{1}{2}S_{k-1}^2(t) - \frac{1}{2}\psi_{k-1}^2(t) \\ &-\frac{1}{2\beta}\int_0^t (\hat{\theta}_k(\tau)-\hat{\theta}_{k-1}(\tau))^T(\hat{\theta}_k(\tau)-\hat{\theta}_{k-1}(\tau))d\tau \\ &\le -\frac{1}{2}S_{k-1}^2(t) \le 0\end{aligned} \tag{33}$$

Step2: *Convergence of tracking error*

In the following we will show the finiteness of $W_0(t)$. From the definition $W_k(t)$ in (22), we have:

$$W_0(t) = \frac{1}{2}S_0^2(t) + \frac{1}{2}\psi_0^2(t) + \frac{1}{2\beta}\int_0^t \phi_0^T(\tau)\phi_0(\tau)d\tau \tag{34}$$

Hence, the derivative of $W_0(t)$ is:

$$\dot{W}_0(t) = S_0(t)\dot{S}_0(t) + \psi_0(t)\dot{\psi}_0(t) + \frac{1}{2\beta}\phi_0^T(t)\phi_0(t) \tag{35}$$

From (20) and (21), it can be derived that:

$$\dot{W}_0(t) \le S_0(t)\phi_0^T(t)\xi(x_0) + \frac{1}{2\beta}\phi_0^T(t)\phi_0(t) \tag{36}$$

From (29) and the fact $\hat{\theta}_{-1} = 0$, we obtained:

$$\begin{aligned}\frac{1}{2\beta}\phi_0^T(t)\phi_0(t) &= \frac{1}{2\beta}\begin{pmatrix}\phi_0^T(t)\phi_0(t) \\ -\phi_{-1}^T(t)\phi_{-1}(t)\end{pmatrix} + \frac{1}{2\beta}\phi_{-1}^T(t)\phi_{-1}(t) \\ &= \frac{1}{\beta}\phi_0^T(t)(\hat{\theta}_0 - \hat{\theta}_{-1}) - \frac{1}{2\beta}\hat{\theta}_0^T\hat{\theta}_0 + \frac{1}{2\beta}\theta^T\theta \\ &\le -S_0(t)\phi_0^T(t)\xi(x_0) + \frac{1}{2\beta}\theta^T\theta\end{aligned} \tag{37}$$

Consequently, from (36) and (37) we have:

$$\dot{W}_0(t) \le \frac{1}{2\beta}\theta^T\theta \tag{38}$$

Because θ is a constant parameter, therefore, $\frac{1}{2}\theta^T\theta$ exists and bounded.

Considering from assumption 3, $S(0) = 0$, then we have:

$$\begin{aligned}W_0(t) &\le \left|W_0(0)\right| + \left|\int_0^t \dot{W}_0(\tau)d\tau\right| \\ &\le \int_0^t \left|\dot{W}_0(\tau)\right|d\tau \\ &\le \int_0^t \frac{1}{2\beta}\theta^T\theta d\tau \\ &\le \frac{t}{2\beta}\theta^T\theta < \infty \;\; \forall t \in [0,T]\end{aligned} \tag{39}$$

From above inequality finiteness of $W_0(t)$ implies and from (33) conclude that $W_k(t) \le W_0(t)$, therefore, $W_k(t)$ is finite.

Note that (33) by using repeatedly, also gives:

$$\begin{aligned}&W_k(t) \le W_0(t) - \frac{1}{2}\sum_{i=1}^{k} S_{k-1}^2(t) \\ &\lim_{k\to\infty} W_k(t) \le W_0(t) - \lim_{k\to\infty}\frac{1}{2}\sum_{i=1}^{k} S_{k-1}^2(t)\end{aligned} \tag{40}$$

Since, $W_0(t)$ is finite and $W_k(t)$ is positive, hence from (40) $\lim_{k\to\infty}\frac{1}{2}\sum_{i=1}^{k} S_{k-1}^2(t)$ and accordingly $\lim_{k\to\infty} S_k^2(t) \;\forall t \in [0,T]$ are convergence. Therefore, $S_k(t)$ converges to zero.

From the result of theorem 2, it concludes that the sliding surface dynamics $S_k(t)$ has convergence to the origin. Since the parameters of sliding surface dynamics is chosen such that it satisfies the stability condition of Lemma, then the output tracking error is convergent and finally the output tracking is satisfied.

5. Simulation Results

In this section, our goal is to achieve ILSM control by applying the method on two different fractional-order systems.

Example 1. Consider the fractional order Chaotic Oscillator, which is written as [27]:

$$\begin{cases} D^{\alpha}x_1(t)=x_2(t) \\ D^{\alpha}x_2(t)=x_3(t) \\ D^{\alpha}x_3(t)=\begin{pmatrix} -a(x_1(t)+x_2(t)+x_3(t)) \\ +\operatorname{sgn}(x_1)+u(t)+d(t) \end{pmatrix} \end{cases} \tag{41}$$

where $\alpha=0.97$, $a=0.4$ and $d(t)=0.1\sin(\pi t)$. The output of above system is considered as follows:

$$y(t)=x_1(t)+x_2(t) \tag{42}$$

It is obvious that this system is assumed partial observability with one unknown parameter.

Regarding (9) and (18), the sliding surface and control law are given:

$$S_k(t)=k_1D^{\alpha-1}e_{1,k}(t)+k_2D^{\alpha-1}e_{2,k}(t) \tag{43}$$

$$\begin{aligned} u_k(t)=D^{\alpha}r_2(t)+k_1e_{2,k}(t)-x_{2,k}(t)-\operatorname{sgn}(x_{1,k}(t)) \\ +\hat{a}_k(x_{1,k}(t)+x_{2,k}(t)+x_{3,k}(t)) \\ -M(\psi_k(t)-1)\operatorname{sgn}(S_k(t)) \end{aligned} \tag{44}$$

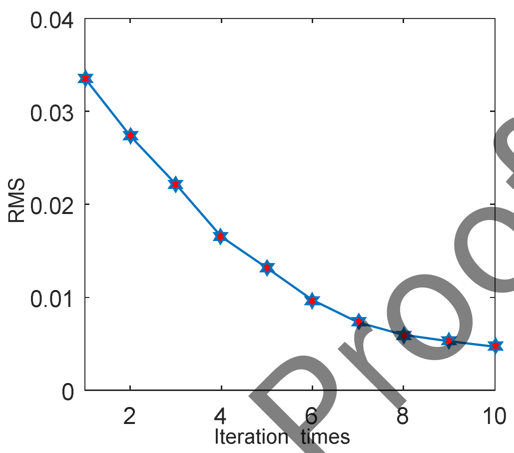

Fig. 3. RMS of output error for 10 iterations

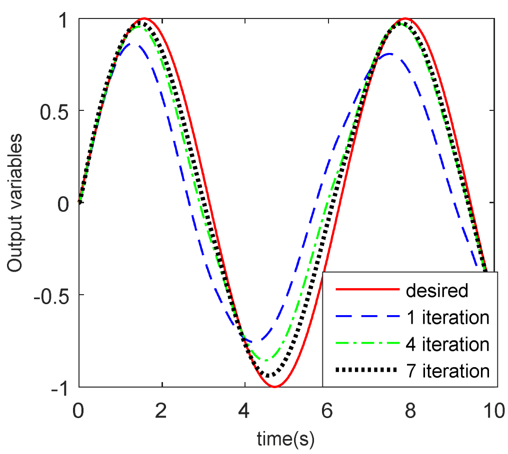

Fig. 4. Output variables for different iterations

where $k_1=0.6$, $k_2=1$, $M=0.8$. For this choice of coefficients, according to lemma and (14) stability of error dynamic will be established. According to (19), IL mechanism is defined:

$$\hat{a}_k(t)=\hat{a}_{k-1}(t)+(x_{1,k}(t)+x_{2,k}(t)+x_{3,k}(t))\beta S_k(t) \tag{45}$$

where $\beta=1$. The reference output is defined $r(t)=\sin(t)$. From assumption 3, the initial values of variable states are in origin.

We operated the Chaotic Oscillator system in 10 iterations. The simulation results are demonstrated in figures 3-6. Fig. 3 shows that the root mean squares (RMS) of the output error, after 10 iterations, gradually tends to zero. In Fig. 4 output variables for different iterations is shown. From this figure, it's obvious that through increasing the number of iteration, process of desired output tracking improves. Fig. 5 displays that the output of system converges to the desired trajectory at the 10th iteration. Fig. 6 indicates that the continuity of the resulting control input signal.

Example 2. Consider the fractional order Duffing system, which is expressed as [28]:

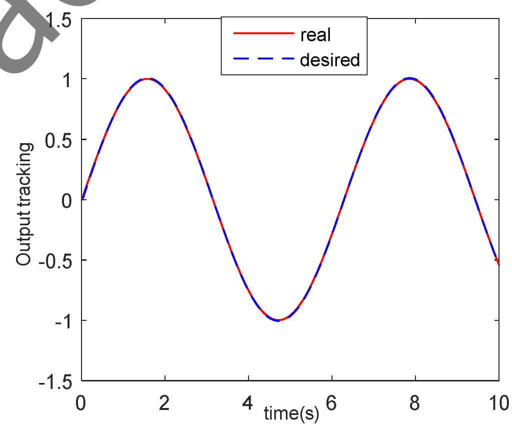

Fig. 5. Real output and the desired trajectory at the10th iteration

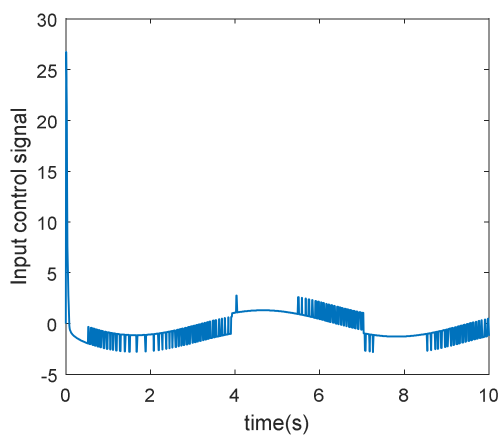

Fig. 6. Control input at the 10th iteration

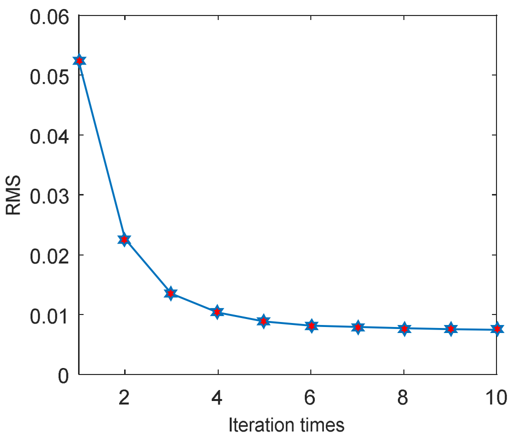

Fig. 7. RMS of output error for 10 iterations

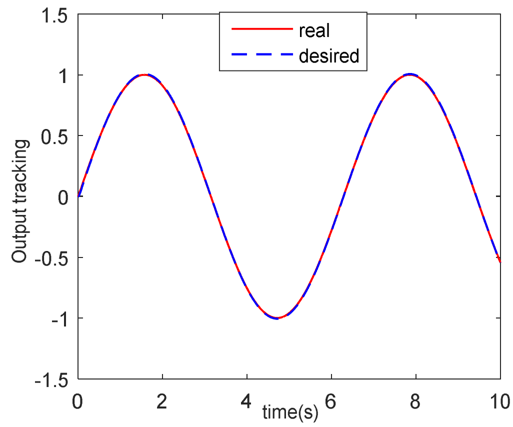

Fig. 9. Real output and the desired trajectory at the 10th iteration

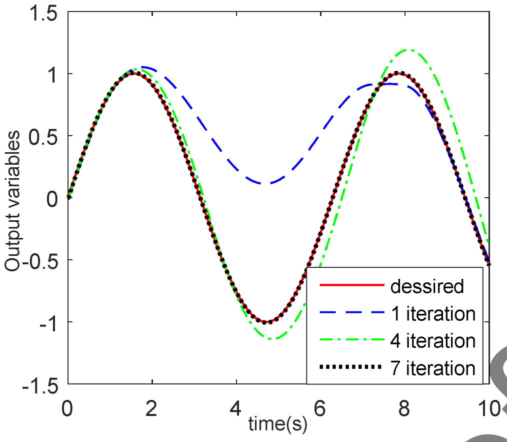

Fig. 8. output variables for different iterations

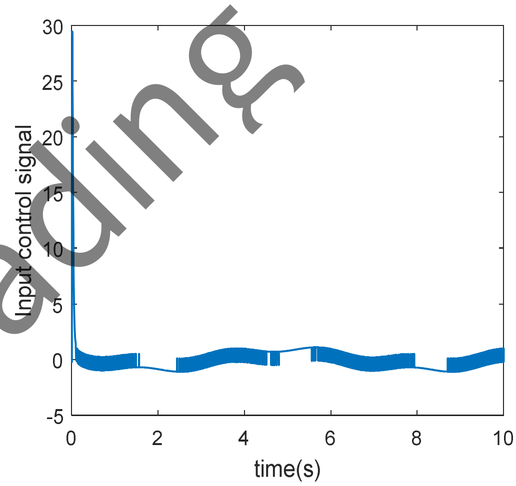

Fig. 10. Control input at the 10th iteration

$$\begin{cases} D^{\alpha}x_1(t) = x_2(t) \\ D^{\alpha}x_2(t) = \begin{pmatrix} -ax_2(t) + bx_1(t) - x_1^3(t) \\ +0.3\cos(t) + u(t) + d(t) \end{pmatrix} \end{cases} \tag{46}$$

where $\alpha = 0.97$, $(a,b) = (0.5,1)$ and $d(t) = 0.1\sin(\pi t)$. The output of above system is considered as follows:

$$y(t) = x_1(t) + x_2(t) \tag{47}$$

As is clear from the above equation the system is assumed complete observability with two unknown parameters.

From (9) and (18), the sliding surface and control law are given:

$$S_k(t) = k_1 D^{\alpha-1} e_{1,k}(t) \tag{48}$$

$$\begin{aligned} u_k(t) = D^{\alpha}r(t) - x_{1,k}(t) + x_{1,k}^3(t) - 0.3\cos(t) \\ -\hat{b}_k(t)x_{1,k}(t) + \hat{a}_k(t)x_{2,k}(t) \\ -M(\psi_k(t)-1)\operatorname{sgn}(S_k(t)) \end{aligned} \tag{49}$$

where $k_1 = 1, M = 0.6$. For this choice of coefficients, according to lemma and (14) stability of error dynamic will be established. According to (19), IL mechanisms are defined:

$$\begin{aligned} \hat{a}_k(t) = \hat{a}_{k-1}(t) + x_{2,k}(t)\beta S_k(t) \\ \hat{b}_k(t) = \hat{b}_{k-1}(t) - x_{1,k}(t)\beta S_k(t) \end{aligned} \tag{50}$$

where $\beta = 0.8$. The reference output is defined $r(t) = \sin(t)$. From assumption 3, the initial values of variable states are in origin.

We operated the Duffing system in 10 iterations. The simulation results are demonstrated in figures 7-10. Fig. 7 shows that the root mean squares (RMS) of the output error, after 10 iterations, gradually tends to zero. In Fig. 8 output variables for different iterations is shown. From this figure, it's obvious that through increasing the number of iteration, process of desired output tracking improves. Fig. 9 displays that the output of system converges to the desired trajectory at the 10th iteration. Fig. 10 indicates that the continuity of the resulting control input signal.

6. Conclusion

In this paper, we propose a new controller with ILSM structure for tracking output linear/nonlinear fractional order systems. Designed controller is a robust controller, mixed time-domain and iteration-domain adaptation law. The integral surface of the sliding mechanism is defined to attenuate the effect of the disturbance. In addition the proposed controller is designed so that without needing to apply it directly to the system output the process of tracking is well performed. A rigorous proof, via composite energy reduction in each iteration, is given to show the finiteness of tuning control parameters, rejection of the random input, disturbance and the asymptotic error convergence along the iteration axis. The simulation results have clearly exhibited the excellent output tracking performance by the proposed robust ILSM controller.

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