A Singular Perturbation-Like Approach to EDFA Gain Control Based on Observer Techniques

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Abstract – In this paper, we propose a singular perturbation-like approach to EDFA gain controller design and analysis. Considering a three-level model of EDFA, a gain controller containing a state observer and a channel add/drop estimator is designed based on a singular perturbation - like concept. The proposed design methodology is shown to be effective and advantageous not only in theoretically verifying the asymptotic stability of systems with multi-time scales such as EDFA but also in designing an asymptotic estimator for channel add/drops which does not satisfy the matching condition.

Keywords: EDFA, Singular perturbation, State observer, Gain control, Channel add/drop estimator

1. Introduction

EDFA(Erbium Doped Fiber Amplifier) is widely used for the amplification of channel signals in a WDM optical network. In an EDFA, it is important to maintain the gain of each channel when channel add/drops or active rearrangements of the network occur. The change of the number of channel signals called a channel add/drop causes a change of the amplifier gain of each channel signal due to the cross gain saturation effect [1].

There have been suggested several methods to handle this issue. One of them uses an EDFA output as feedback signal in an optical feedback control loop [2]. It, however, has the drawback that the frequency of channel add/drops should be less than that of the relaxation oscillation frequency of the EDFA, which is several hundred hertz. On the other hand, the gain fluctuation due to channel add/drops can be effectively compensated for by controlling the pump laser output electrically according to the EDFA output signal level [3]. In the previous papers [4-7], we proposed a novel technique which minimizes the gain-transient time effectively under the assumption that the rate of Erbium ions at level 3 converges relatively fast to the desired equilibrium compared with the one at level 2. A simplified two-level EDFA model was considered to design a gain controller and a disturbance observer (DOB) technique [8,9,10], and a proportional /integral (PI) controller was applied to the control of the EDFA gain in WDM add/drop networks. However, in order to compensate for the gain fluctuation due to channel add/drops as fast as in the order of micro-seconds, a full three-level model should be considered and a nominal gain controller should be designed considering the state of the population of Erbium ions at level 3. In a simplified two-level model, the matching condition is satisfied and channel add/drops can be easily controlled by a disturbance observer. However, the matching condition is not satisfied by the three-level model, so a new EDFA gain controller design methodology based on the three level model is necessary. In [5], a PID gain control algorithm considering the three-level EDFA model was applied to a nominal control. Since a channel add/drop compensator was still designed using a DOB based on a simplified two-level model, theoretical analysis of asymptotic stability could not be provided rigorously.

In this paper, a theoretical design and analysis of EDFA gain control system is carried out based on a mathematical three level EDFA model [11] using a singular perturbation technique [12]. In order to compensate for channel add/drop effects, a channel add/drop estimator is designed based on an internal model of EDFA, and an EDFA gain controller is proposed combining a state observer with the channel add/drop estimator. With successive applications of time scale separation to the designed EDFA control system, a singular perturbation technique gives a theoretical performance analysis of the proposed EDFA gain control algorithm even in the case that the matching condition is not satisfied. Through simulations, the practicality of the proposed control algorithm is also confirmed.

2. Design of EDFA Gain Control System

2.1 Three-level EDFA Model

In order to design an EDFA gain controller, the following
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three-level model is considered [11]. The energy level of EDF A is shown in Fig. 1 and the equations for the three-level process are given as:

\[
\frac{dN_1}{dt} = -\Gamma_{32}N_3 - (N_3\sigma^s_p - N_1\sigma^s_p)\phi_p
\]
\[
\frac{dN_2}{dt} = -\Gamma_{21}N_2 + (N_1\sigma^s_p - N_2\sigma^s_p)\phi_p + \Gamma_{32}N_3
\]
\[
\frac{dN_3}{dt} = \Gamma_{21}N_1 - (N_1\sigma^s_e - N_2\sigma^s_e)\phi_e + (N_3\sigma^s_p - N_1\sigma^s_p)\phi_p
\]

Where \( \Gamma_{21}, \Gamma_{32} \) are positive constants; \( \phi_e, \phi_p \) are photon flux densities per second of the signal and the pump; \( \sigma^s_e, \sigma^s_p, \sigma^s_a \) are absorption and emission cross section of the signal and the pump (\( \sigma^f = \sigma^s + \sigma^a \)); and \( N_1, N_2, N_3 \) are the number of erbium-ions at each energy level (\( N=N_1+N_2+N_3=1 \)). The power \( P_s \) of the signal and the power \( P_p \) of the pump obey the following equations:

\[
\frac{dP_s}{dt} = \rho\Gamma_p\left(\sigma^e_pN_3 + \sigma^a_pN_2 - \sigma^a_e\right)P_p
\]
\[
\frac{dP_p}{dt} = \rho\Gamma_s\left(\sigma^e_sN_2 + \sigma^a_sN_3 - \sigma^a_e\right)P_s
\]

where \( \rho \) is the Erbium density, and \( \Gamma_s \) and \( \Gamma_p \) are respectively the geometric correction factor for the overlap between the power and the erbium-ions.

Define a reservoir \( r_i(t), i = 2, 3 \) that represents the number of excited Erbium-ions at each level and the EDF A gain of the \( k \)-th channel as follows:

\[
r_i(t) = \rho A \int_0^L N_i(z, t) dz, i = 2, 3
\]
\[
G_k(t) = \ln\frac{P^\text{out}}{P^\text{in}}
\]

integrating (1)-(3) along the whole length of EDF A, we can obtain the following three-level EDF A model equations from definitions of reservoir \( r_i(t), i = 2, 3 \) and \( G_k(t) = \ln(P^\text{out}/P^\text{in}) \):

\[
\frac{dr_2}{dt} = -\Gamma_{32}r_3 + \left(1 - e^{G_p(t)}\right)P^\text{in}_p(t)
\]
\[
\frac{dr_3}{dt} = -\Gamma_{21}r_2 + \Gamma_{32}r_3 + \sum_{k=1}^N \left(1 - e^{G_k(t)}\right)P^\text{in}_k
\]
\[
G_k(t) = B_kr_k - A_k
\]

where \( N \) is the number of channels, \( G_p(t) \) is the gain of input pump channel and

\[
\phi_p = \Gamma_p\frac{P_p}{A}, \phi_e = \Gamma_s\frac{P_s}{A}, A_k = \rho \Gamma_k \sigma^a_e L, B_k = \frac{\Gamma_k}{A}
\]

Suppose that the \( k \)-th channel gain \( G_k(t) \) should be maintained to be a desired constant channel gain \( G^\text{C}_k \). Then, the state variable \( r_2 \) in the EDF A model Eqs. (8) and (9) must satisfy

\[
r_2 = r^*_2 = \frac{1}{B_k}(G^\text{C}_k + A_k)
\]

at the steady state or equilibrium. Define an error variable as

\[
e_2 = r_2 - r^*_2.
\]

Then, the error dynamics are written as

\[
\frac{dr_2}{dt} = -\Gamma_{32}r_3 + \left(1 - e^{G_p(t)}\right)P^\text{in}_p(t)
\]
\[
\frac{de_2}{dt} = -\Gamma_{21}e_2 + \Gamma_{32}r_2 - \Gamma_{32}r^*_2 + \sum_{k=1}^N \left(1 - e^{G_k(t)}\right)P^\text{in}_k
\]

Our goal is to design a stabilizing controller for the system described by (14) and (15). The term \( \sum_{k=1}^N \left(1 - e^{G_k(t)}\right)P^\text{in}_k \) consists of channel signals and varies according to channel add/drops which is not predictable in advance and is considered as a disturbance. So a disturbance observer technique can be adopted to reject the influence of channel add/drops on the channel gain variation. However, the control input \( P^\text{in}_k \) does not appear in the same equation with this term and thus it is nontrivial to compensate for this channel add/drops. The system (14) and (15) does not satisfy the so-called matching condition. In order to overcome this difficulty, we employ a singular perturbation method. If the dynamics (14) can be made much faster than
the dynamics (15) by a control, a singular perturbation can be applied and we can reduce the dynamics such that the reduced dynamics satisfy the matching condition. We can then design a stabilizing controller for this reduced dynamic system using error state feedback and a disturbance estimator.

2.2 EDFA gain controller

In order to design a stabilizing controller for the system in (14) and (15), let us make the following assumption:

(A1) The gain \( G_p(t) \) of the input pump channel is measurable.

Then, a stabilizing controller for the error dynamics (14) and (15) is designed as follows.

\[
P_p^m(t) = -\frac{1}{1-e^{G_p(t)}} \left[ k_3\hat{e}_2(t) + k_2\hat{r}_2(t) + k_3\hat{c}(t) - \frac{\Gamma_{32}}{\Gamma_{32} + k_2}r_2^2 \right]
\]

where \( \hat{r}_2(t) \) and \( \hat{c}(t) \) are respectively estimation variables of the state \( r_2(t) \) and the disturbance \( c(t) \) = \( \sum_{k=1}^{N}(1-e^{G_k(t)})P_k^m \) and \( k_i, i = 1, 2, 3 \) are positive constants. In (16), the term \( k_2\hat{r}_2(t) \) is to make the dynamics in (14) much faster than the one in (15). Since \( r_2(t) \) is not measurable, we use its estimated value \( \hat{r}_2(t) \) instead. The term \( k_3\hat{c}(t) \) is to reject the term \( c(t) \). So we need to design a state estimator for \( r_2(t) \) and a disturbance observer for \( c(t) \).

2.3 Design of a channel add/Drop estimator

As mentioned in the previous section, we need to estimate the term \( \sum_{k=1}^{N}(1-e^{G_k(t)})P_k^m \) in (15) including channel add/drops. It usually costs a lot to measure all the channel powers \( P_k^m \) and all the channel gains \( G_k \) of the EDFA in optical networks. So it is inevitable to estimate the term \( \sum_{k=1}^{N}(1-e^{G_k(t)})P_k^m \). In order to estimate this, we consider the following internal nominal model of the EDFA:

\[
\frac{d\hat{r}_3}{dt} = -\Gamma_{32}\hat{r}_3 + (1-e^{G_p(t)})P_p^m(t)
\]

\[
\frac{d\hat{r}_2}{dt} = -\Gamma_{21}\hat{r}_2 + \Gamma_{32}\hat{r}_3
\]

Define

\[
\hat{G}_k(t) = B_k\hat{r}_2 - A_k.
\]

\[
\hat{e}_2(t) = r_2(t) - \hat{r}_2(t)
\]

\[
\hat{e}_3(t) = r_3(t) - \hat{r}_3(t)
\]

\[
\hat{c}(t) = \sum_{k=1}^{N}(1-e^{G_k(t)})P_k^m.
\]

Then, we obtain the following equations:

\[
\frac{d\hat{e}_3}{dt} = -\Gamma_{32}\hat{e}_3
\]

\[
\frac{d\hat{e}_2}{dt} = -\Gamma_{21}\hat{e}_2 + \Gamma_{32}\hat{e}_3 + c(t)
\]

\[
\hat{e}_2(t) = B_k\hat{e}_2.
\]

Notice that the system (21)-(23) has stable zero dynamics. So we have the following transfer function between the channel add/drop input \( c(t) = \sum_{k=1}^{N}(1-e^{G_k(t)})P_k^m \) and the output \( \hat{e}_G(t) \):

\[
P(s) = \frac{L[\hat{G}_k(t)]}{L[c(t)]} = \frac{B_k}{s + \Gamma_{21}}
\]

where \( L[\cdot] \) denotes the Laplace transform. Define a filter \( Q(s) \) by

\[
Q(s) = \frac{L[\hat{c}(t)]}{L[\hat{e}_G(t)]} = \frac{A_D(s + \Gamma_{21})}{B_k(s + A_D)}
\]

where \( \hat{c}(t) \) is the estimated output of \( c(t) \). We have the following relation between the channel add/drop signal \( c(t) \) and its estimate \( \hat{c}(t) \):

\[
\frac{L[\hat{c}(t)]}{L[c(t)]} = \frac{A_D}{s + A_D}
\]

where the positive constant \( A_D \) is to be chosen later. Here we use a first-order linear model for the resultant channel add/drop estimator for convenience, but any higher order model can be equally used.

2.4 Design of a state observer

In order to stabilize the error system given by (14) and (15) with error state feedback, we need to estimate the state variable \( r_2(t) \). Usually a state estimator can be easily designed if the system is observable, but its design
becomes nontrivial when the term $\sum_{k=1}^{N}(1-e^{G_k(t)})P^0_k$ in the EDFA model given by (8)-(10) cannot be measured. However, it is possible to design a state estimator that guarantees asymptotic estimation performance even when an unknown term $\sum_{k=1}^{N}(1-e^{G_k(t)})P^0_k$ is present, if we use the channel add/drop estimator proposed in the previous section. Now we propose the following state estimator for the system (8)-(10):

$$\dot{\hat{G}}_k(t) = B_k \hat{r}_k - A_k$$  \hspace{1cm} (29)

where $L_1$ and $L_2$ are observer gains. The observer gains $L_1$ and $L_2$ are chosen such that $[\begin{array}{c} -\Gamma_{32} \\ L_4 B_k \\ \Gamma_{32} \\ -\Gamma_{32} - L_2 B_k \end{array}]$ is stable.

### 3. Theoretical Analysis : A Singular Perturbation Approach

In this section, we introduce a singular perturbation approach to stability analysis, which provides a systematic procedure for analysis of multi-time scaled systems.

#### 3.1 Reduced dynamics of time-scaled closed-loop system

Since the estimator should have a faster performance than the controller, the control system designed in the previous section is considered as a multi-time scaled system. So a singular perturbation method can be applied to the analysis of the EDFA gain control system designed in Section II.

From (14) - (16) and (27) - (29), we obtain the error equations of the closed loop system as follows:

$$\Sigma_1: \frac{d\hat{r}_1}{dt} = -(\Gamma_{32} + k_2) \begin{pmatrix} r_1 - \frac{\Gamma_{32}}{\Gamma_{32}} \end{pmatrix} - k_1 e_2(t) + k_2 \hat{e}_3 - k_2 \hat{c}(t)$$  \hspace{1cm} (30)

$$\frac{de}{dt} = -\Gamma_{21} e_2 + \Gamma_{32} \hat{r}_3 - \Gamma_{21} \hat{r}_2 + c(t)$$  \hspace{1cm} (31)

$$\frac{d\hat{e}}{dt} = -\Gamma_{32} \hat{e}_3 + L_4 B_k \hat{e}_2$$  \hspace{1cm} (32)

$$\frac{d\hat{c}}{dt} = -\Gamma_{21} \hat{c}_2 + \Gamma_{32} \hat{e}_3 + [c(t) - \hat{c}(t)] + L_2 B_k \hat{e}_2$$  \hspace{1cm} (33)

where

$$e_2 = r_2 - r_2^*, \quad \hat{e}_2(t) = r_2(t) - \hat{r}_2(t),$$

$$\hat{e}_3(t) = r_3(t) - \hat{r}_3(t).$$  \hspace{1cm} (34)

From (26), we obtain the following equation for the channel add/drop estimator:

$$\frac{d\hat{c}(t)}{dt} = -A_D \left[ \hat{c}(t) - c(t) \right].$$  \hspace{1cm} (35)

If the design parameter $A_D$ in (35) is chosen so that the dynamics given in (35) is faster than any other subsystems (30)-(33), then the system (35) is stabilized very fast and $\hat{c}(t)$ immediately converges to $c(t)$.

Since $k_2$ is chosen so that $(\Gamma_{32} + k_2)$ becomes much larger than $\Gamma_{21}$, a singular perturbation procedure is applied to (30)-(35) by letting $r_3 \to 0$. Let $k_3$ be chosen as

$$k_3 = \frac{\Gamma_{32} + k_2}{\Gamma_{32}}.$$  \hspace{1cm} (36)

Then we obtain the following reduced dynamics.

$$\Sigma_2: \frac{d\hat{r}_1}{dt} = -\left( \Gamma_{21} + \frac{\Gamma_{32} k_1}{\Gamma_{32} + k_2} \right) r_2^*$$

$$+ \frac{\Gamma_{32} k_2}{\Gamma_{32} + k_2} \hat{e}_3 + \hat{c}(t) - \hat{c}(t)$$

$$\frac{d\hat{e}_2}{dt} = -\Gamma_{32} \hat{e}_3 + L_4 B_k \hat{e}_2$$  \hspace{1cm} (38)

$$\frac{d\hat{e}_3}{dt} = -\Gamma_{21} \hat{e}_2 + \Gamma_{32} \hat{e}_3 + [c(t) - \hat{c}(t)] + L_2 B_k \hat{e}_2$$  \hspace{1cm} (39)

$$\frac{d\hat{c}}{dt} = -A_D \left[ \hat{c}(t) - \hat{c}(t) \right].$$  \hspace{1cm} (40)

where

$$\hat{c}(t) = \sum_{k=1}^{N}(1-e^{G_k(t)})P^0_k$$  \hspace{1cm} (41)

and

$$\hat{r}_2 = r_2 - r_2^*.$$  \hspace{1cm} (42)

and $\hat{r}_2$ and $\hat{G}_k$ satisfy the following:

$$\frac{dr_2}{dt} = -\left( \Gamma_{21} + \frac{\Gamma_{32} k_1}{\Gamma_{32} + k_2} \right) (r_2 - r_2^*) + \frac{\Gamma_{32} k_2}{\Gamma_{32} + k_2} \hat{e}_3 + \hat{c}(t) - \hat{c}(t)$$

$$\hat{G}_k(t) = B_k \hat{r}_2 - A_k.$$  \hspace{1cm} (43)

The design parameter $A_D$ in (35) or (40) is chosen so that the dynamics given by (35) or (40) is faster than the
other subsystem dynamics (37)-(39). Then the system (40) is stabilized very fast and \( \dot{X}(t) \) immediately converges to \( X(t) \). So, if we apply a singular perturbation method again to (37)-(40) as \( \dot{X}(t) \to 0 \), we have the following reduced dynamics:

\[
\begin{align*}
\Sigma_3: & \frac{d\hat{e}_2}{dt} = -\left( \Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32} + k_2} \right) \hat{e}_2 - \frac{\Gamma_{32}k_2}{\Gamma_{32} + k_2} \hat{e}_3, \\
\frac{d\hat{e}_3}{dt} &= -\Gamma_{32}\hat{e}_3 + L_e B_k \hat{e}_2, \\
\frac{d\hat{e}_2}{dt} &= -\Gamma_{21}\hat{e}_2 + \frac{\Gamma_{32}k_1}{\Gamma_{32} + k_2} \hat{e}_2 + \frac{\Gamma_{32}k_2}{\Gamma_{32} + k_2} \hat{e}_3.
\end{align*}
\]  
(44)

Since the design parameters \( L_1 \) and \( L_2 \) of the state estimator are designed so that its performance is much faster than error state feedback control and stable, the estimation error states \( \hat{e}_2 \) and \( \hat{e}_3 \) of the reduced dynamics in (45) and (46) decay to zero much faster than \( \hat{e}_2 \). So, as \( \hat{e}_2(t) \to 0, \hat{e}_3(t) \to 0 \), we have the following reduced dynamics for the system \( \Sigma_3 \):

\[
\Sigma_4: \frac{d\hat{e}_2}{dt} = -\left( \Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32} + k_2} \right) \hat{e}_2.
\]  
(47)

From (47), it is obvious that

\[
\left| \hat{e}_2(t) \right| \leq \left| \hat{e}_2(0) \right| e^{-\lambda t}, \forall t \geq 0
\]  
(48)

where

\[
\lambda = \Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32} + k_2}.
\]  
(49)

The performance of the control system is determined by a desired bandwidth \( \lambda \), and the controller gain \( k_1 \) is chosen as

\[
k_1 = \frac{\Gamma_{32} + k_2}{\Gamma_{32}} (\lambda - \Gamma_{21})
\]  
(50)

for given \( \lambda \) and \( k_2 \). Choice of the controller gain \( k_2 \) is discussed in the next section.

### 3.2 Stability analysis

Stability analysis is carried out by showing the asymptotic stability of each system \( \Sigma_i, i = 1, 2, 3 \) successively using asymptotic stability of systems \( \Sigma_m, m = i + 1, \ldots, 4 \). In order to show the asymptotic stability, we assume that channel add/drops are not persistent. That is, channel add/drops are assumed to occur finitely many times. We make the following assumption.

(A2) The number of Channel add/drops, \( M \), is finite.

Define the instants at which channel add/drops occur by a time sequence \( t_n, n = 1, \ldots, M \). So, if we define \( t_{M+1} = \infty \), each channel input \( p_k^\mu, k = 1, \ldots, N \) is zero or a positive constant for all \( t \in (t_i, t_{i+1}) \), \( i = 1, \ldots, M \). So, the time derivative of each channel input is zero for all \( t \in (t_i, t_{i+1}) \), \( i = 1, \ldots, M \) and the following equation holds for any \( \epsilon > 0 \):

\[
\frac{d}{dt} p_k^\mu(\epsilon, t_i) = 0, k = 1, \ldots, N, \forall t \in (t_i, t_{i+1}), i = 1, \ldots, M.
\]  
(51)

### Step 1. Stability analysis of \( \Sigma_3 \)

In order to show that the system \( \Sigma_3 \) is asymptotically stable, we define the error variables between the system \( \Sigma_3 \) and the system \( \Sigma_4 \) as follows:

\[
E(t) = [\hat{e}_2 \: \hat{e}_3 \: \hat{e}_3]^T, \quad E_2 = \hat{e}_2 - \bar{e}_2.
\]  
(52)

Then,

\[
\frac{dE_2}{dt} = -\left( \Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32} + k_2} \right) E_2 - \frac{\Gamma_{32}k_2}{\Gamma_{32} + k_2} E_3
\]  
(53)

\[
\frac{dE_3}{dt} = -\Gamma_{32} E_3 + L_e B_k \hat{e}_2
\]  
(54)

\[
\frac{dE_2}{dt} = -\Gamma_{21} E_2 + \frac{\Gamma_{32}k_1}{\Gamma_{32} + k_2} E_2 + \frac{\Gamma_{32}k_2}{\Gamma_{32} + k_2} E_3.
\]  
(55)

It is obvious that the system (53)-(55) is asymptotically stable. So, there exist positive numbers \( \alpha_3 \) and \( \beta_3 \) such that

\[
\|E(t)\| \leq \alpha_3 \|E(0)\| e^{-\beta_3 t}, \forall t \geq 0.
\]  
(56)

### Step 2. Stability analysis of \( \Sigma_4 \)

Define an error vector \( \bar{E}_2 \) using the asymptotic stability of \( \Sigma_3 \) and \( \Sigma_4 \):

\[
\bar{E}_2(t) = \begin{bmatrix} \hat{e}_2(t) \\ \hat{e}_3(t) \\ \hat{e}_3(t) \end{bmatrix}, \quad \hat{e}_2(t) = \hat{e}_2 - \bar{e}_2.
\]  
(57)

\[
\bar{E}_3(t) = \bar{E}_3(t) - \hat{e}_3(t), \quad \hat{e}_3(t) = \hat{e}_3(t) - \hat{e}_3(t),
\]

\[
\hat{e}_3(t) = \hat{e}_3(t) - \hat{e}_3(t).
\]

From (37)-(40) and (44)-(46), the error system is described as follows:

\[
\frac{d}{dt} \bar{E}_2 = -\left( \Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32} + k_2} \right) \bar{E}_2 + \frac{\Gamma_{32}k_2}{\Gamma_{32} + k_2} \hat{e}_3 + \bar{E}_c
\]  
(58)
\[
\frac{d}{dt} \hat{E}_2 = -(\Gamma_{21} - L_2 B_k) \hat{E}_2 + \Gamma_{32} \hat{E}_3 + \hat{E}_c
\]

(60)

Since the length of EDF, \( l \) is finite, the reservoir \( r_2 \), \( G_k(t) \) and \( \bar{c}(t) \) defined in (41) are bounded. So, \( \bar{E}_c \) is also bounded from (40) such that there exists a positive constant \( B_c \) satisfying

\[
|\bar{E}_c| < B_c, \forall t \geq 0.
\]

(61)

Therefore, from (57), (58), (59), and (61), there exists a positive constant \( B_E \) such that the following inequality holds.

\[
\left\| \bar{E}(t) \right\| \leq B_E, \forall t \geq 0
\]

(62)

From (41), (43), (51) and (62),

\[
\frac{d}{dt} \bar{c}(t) = \frac{d}{dt} \left( \sum_{k=1}^{N} (1 - e^{G_k(t)}) P_k^{a_n} \right) = \sum_{k=1}^{N} \left( -e^{G_k(t)} \bar{G}_k(t) P_k^{a_n} \right)
\]

\[
= \sum_{k=1}^{N} \left( -e^{G_k(t)} B_k P_k^{a_n} \right) \left( \bar{r}_c(t) \right)
\]

(63)

and

\[
\frac{d}{dt} \bar{r}_c = -\left( \Gamma_{21} + \frac{\Gamma_{32} k_1 }{\Gamma_{32} + k_2} \right) (\bar{r}_c - \bar{r}_c^*) + \frac{\Gamma_{32} k_2}{\Gamma_{32} + k_2} \bar{r}_c + \bar{E}_c
\]

(64)

Thus, for all \( t \in [t_i, +\infty), i = 1, \cdots, M \),

\[
\frac{d}{dt} \bar{E}_c = -A \bar{E}_c + \bar{c}
\]

\[
= -(A_0 + \Pi_1) \bar{E}_c + \Pi_1 \left( \Gamma_{21} + \frac{\Gamma_{32} k_1 }{\Gamma_{32} + k_2} \right) (\bar{E}_2 + \bar{E}_c + \bar{r}_c)
\]

\[-\Pi_1 \left( \Gamma_{32} k_2 \left( \bar{E}_3 + \bar{r}_c \right) \right)
\]

(65)

where

\[
\Pi_1(t) = \sum_{k=1}^{N} B_k e^{G_k(t)} P_k^{a_n}.
\]

(66)

Meanwhile, the reservoir \( r_2 \) is bounded and \( G_k(t) \) is also bounded since the length of EDF, \( l \) is finite. So, there exist positive constants \( \Pi_m \) and \( \Pi_M \) such that

\[
\Pi_m \leq \Pi_1(t) \leq \Pi_M, \forall t \geq 0.
\]

(67)

Define a matrix \( A_3 \) by

\[
A_3 = \begin{bmatrix}
\Gamma_{21} + \frac{\Gamma_{32} k_1 }{\Gamma_{32} + k_2} & \frac{\Gamma_{32} k_2 }{\Gamma_{32} + k_2} & 0 \\
0 & -\Gamma_{32} & L_2 B_k \\
0 & \Gamma_{32} & -(\Gamma_{21} - L_2 B_k)
\end{bmatrix}
\]

(68)

Since \( A_3 \) is stable for any positive numbers \( k_1 \) and \( k_2 \), there exist positive definite symmetric matrices \( P \) and \( Q \) satisfying the following Lyapunov equation

\[
A_3^T P + P A_3 = -Q.
\]

(69)

Define a Lyapunov-like function \( V_j \) by

\[
V_j(t) = X_j^T P_j X_j
\]

(70)

where

\[
X_j = \begin{bmatrix}
\bar{E}^T \\
\bar{E}_c
\end{bmatrix}, P_j = \text{diag}(P, \rho_c), \rho_c > 0.
\]

(71)

Then, from (58) and (65), for all \( t \in [t_i, +\infty), \)

\[
\frac{d}{dt} V_j(t) = \bar{E}^T (A_3^T P_j + P_j A_3) \bar{E} + 2\bar{E}^T P_j \bar{E}_c
\]

\[-2\rho_c \left( \bar{A}_0 + \Pi_1 \right) \bar{E}_c - 2\rho_c \Pi_1 \left( \frac{\Gamma_{32} k_2 }{\Gamma_{32} + k_2} \right) \left( \bar{E}_2 + \bar{E}_c + \bar{r}_c \right) \bar{E}_c
\]

\[+2\rho_c \Pi_1 \Gamma_{21} + \frac{\Gamma_{32} k_1 }{\Gamma_{32} + k_2} \left( \bar{E}_2 + \bar{E}_c + \bar{r}_c \right) \bar{E}_c
\]

(72)

where \( B_k = [1 1 0]^T \). Since \( \Pi_1(t) \) is positive and bounded as in (67), we have the following inequality for all \( t \in [t_i, +\infty) \).

\[
\frac{d}{dt} V_j(t) \leq -\bar{E}^T Q \bar{E} + 2\bar{E}^T P_j \bar{E}_c
\]

\[-2\rho_c \left( \bar{A}_0 + \Pi_1 \right) \bar{E}_c - 2\rho_c \Pi_1 \left( \frac{\Gamma_{32} k_2 }{\Gamma_{32} + k_2} \right) \left( \bar{E}_2 + \bar{E}_c + \bar{r}_c \right) \bar{E}_c
\]

\[+2\rho_c \Pi_1 \Gamma_{21} + \frac{\Gamma_{32} k_1 }{\Gamma_{32} + k_2} \left( \bar{E}_2 + \bar{E}_c + \bar{r}_c \right) \bar{E}_c
\]

(73)

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\( \leq -X_3^T Q(t) X_3 + 2 \rho c \Pi_M \left( \Gamma_{21} + \frac{\Gamma_{32} k_1}{\Gamma_{32} + k_2} \right) \left| E_2 \right| \left| E_2 \right| \)
\[ + 2 \rho c \Pi_M \left( \frac{\Gamma_{32} k_2}{\Gamma_{32} + k_2} \right) \left| \tilde{E}_3 \right| \left| \tilde{E}_3 \right| \] (73)

where

\[ \tilde{Q}(t) = \begin{bmatrix} \lambda_m(Q(t)) I_{3 \times 3} & -PB_c + \Pi_1(t) \Lambda' \\
(-PB_c + \Pi_1(t) \Lambda)' & 2 \rho c (A_D + \Pi_m) \end{bmatrix}, \]
\[ \Lambda = [q_{14} \ q_{24} \ 0]^T, \quad q_{14} = -2 \rho c \left( \Gamma_{21} + \frac{\Gamma_{32} k_1}{\Gamma_{32} + k_2} \right), \quad q_{24} = 2 \rho c \frac{\Gamma_{32} k_2}{\Gamma_{32} + k_2}. \] (74)

For a positive number \( 0 < \bar{\lambda} < \lambda_m(Q) \), choose a design parameter \( A_D \) as follows
\[ A_D > \frac{1}{2 \rho c} \left( \bar{\lambda} + \frac{\delta}{\lambda_m(Q) - \bar{\lambda}} \right) - \Pi_m \] (75)

where

\[ \delta = \max_{\Pi_1(t)} \left\{ (PB_c - \Pi_1(t) \Lambda)' (PB_c - \Pi_1(t) \Lambda) \right\}. \] (76)

From (67), it follows that

\[ \delta = \max_{\Pi_1(t)} \left\{ (PB_c - \Pi_1(t) \Lambda)' (PB_c - \Pi_1(t) \Lambda) \right\} \]
\[ = \max_{\Pi_1(t)} \left\{ (PB_c - \Pi_1(t) \Lambda)' (PB_c - \Pi_1(t) \Lambda) \right\} \] (77)

where

\[ \delta_m = (PB_c - \Pi_1(t) \Lambda)' (PB_c - \Pi_1(t) \Lambda), \]
\[ \delta_M = (PB_c - \Pi_1(t) \Lambda)' (PB_c - \Pi_1(t) \Lambda). \] (78)

Then, \( \bar{Q} \) is positive definite and satisfies

\[ \bar{Q}(t) > \bar{\lambda} I_{4 \times 4}. \] (79)

So, the Lyapunov function \( V_3 \) satisfies the following inequality for all \( t \in [t_i + \epsilon, t_{i+1}] \), \( i = 1, \cdots, M \).
\[ \frac{d}{dt} V_3(t) \leq -\frac{\lambda_m(\bar{Q}(t))}{\lambda_m(P)} V_3(t) + 2 \rho c \Pi_M \left( \Gamma_{21} + \frac{\Gamma_{32} k_1}{\Gamma_{32} + k_2} \right) \left| E_2 \right| \left| E_2 \right| \]
\[ + 2 \rho c \Pi_M \left( \frac{\Gamma_{32} k_2}{\Gamma_{32} + k_2} \right) \left| \tilde{E}_3 \right| \left| \tilde{E}_3 \right| \] (80)

where \( \lambda_m(\cdot) \) and \( \lambda_M(\cdot) \) are respectively the smallest and the largest eigenvalue of the associated matrix. Dividing (80) by \( \frac{1}{V_3(t)} \) leads to the following inequality:

\[ \frac{d}{dt} \frac{1}{V_3(t)} \leq -\frac{\bar{\lambda}}{2 \lambda_m(P)} \frac{1}{V_3(t)} + \rho c \Pi_M \left( \frac{\Gamma_{32} k_1}{\Gamma_{32} + k_2} \right) \left| E_2 \right| \left| E_2 \right| \]
\[ + \rho c \Pi_M \left( \frac{\Gamma_{32} k_2}{\Gamma_{32} + k_2} \right) \left| \tilde{E}_3 \right| \left| \tilde{E}_3 \right| \] (81)

for all \( t \in [t_i + \epsilon, t_{i+1}] \). It follows from (48), (56), and (81) that

\[ \frac{1}{V_3(t)} \leq \frac{1}{V_3(t_i + \epsilon)} e^{-\frac{\bar{\lambda}}{2 \lambda_m(P)}(t_i - \epsilon - \epsilon)} \]
\[ + \rho c \Pi_M \left( \frac{\Gamma_{32} k_1}{\Gamma_{32} + k_2} \right) \left| E_2 \right| \left| E_2 \right| \]
\[ + \rho c \Pi_M \left( \frac{\Gamma_{32} k_2}{\Gamma_{32} + k_2} \right) \left| \tilde{E}_3 \right| \left| \tilde{E}_3 \right| \] (82)

for all \( t \in [t_i + \epsilon, t_{i+1}] \). So, there exist positive numbers \( \bar{\alpha}_{3,i} \) and \( \bar{\beta}_{3,i} \) for each \( i \in [1, M] \) such that

\[ \| X_3(t) \| \leq \bar{\alpha}_{3,i} \| X_3(t) \| e^{-\bar{\beta}_{3,i}(t_i - \epsilon - \epsilon)}, \]
\[ \forall t \in [t_i + \epsilon, t_{i+1}] \] (83)

From (70) and (71), there exists a positive constant \( B_{X3} \) such that the following inequality holds.

\[ \| X_3(t) \| \leq B_{X3}, \forall t \geq 0 \] (84)

Therefore, the asymptotic stability is satisfied since (83) holds for \( t \in [t_M + \epsilon, \infty] \).

Step 3. Stability Analysis of \( \Sigma_4 \) using the stability of \( \Sigma_2 \), \( \Sigma_3 \), and \( \Sigma_4 \).

Define an error vector \( \tilde{E} \) and an error variable \( \tilde{E}_3 \) by
\[ \tilde{E}(t) = \begin{bmatrix} \tilde{E}_1(t) \\
\tilde{E}_2(t) \\
\tilde{E}_3(t) \\
\tilde{E}_4(t) \end{bmatrix}, \]
\[ \tilde{E}_3(t) = r_3(t) - \bar{r}_3(t) \] (85)

where
Then, we have the following error equations:

\[
\begin{align*}
\frac{d\hat{E}_3}{dt} &= -(\Gamma_{32} + k_2)\hat{E}_3 + \frac{1}{\Gamma_{32} + k_2} \left[k_1\hat{e}_2(t) - k_2\hat{e}_3(t) + k_3\hat{c}(t)\right], \\
\frac{d\hat{E}_2}{dt} &= -\left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32} + k_2}\right)\hat{E}_2 + \Gamma_{32}\hat{E}_3 + \frac{\Gamma_{32}k_2}{\Gamma_{32} + k_2} \hat{\tilde{c}}(t) \\
&\quad + \hat{E}_C - \hat{E}_C + \frac{\Gamma_{32}k_2}{\Gamma_{32} + k_2} \hat{\tilde{c}}(t) \\
\frac{d\hat{E}_1}{dt} &= -\Gamma_{32}\hat{E}_3 + L_1B_i\hat{E}_2 \\
\frac{d\hat{E}_2}{dt} &= -\left(\Gamma_{21} + L_2B_i\right)\hat{E}_2 + \Gamma_{32}\hat{E}_3 + \hat{E}_C - \hat{E}_C.
\end{align*}
\]

Rewriting (31) and (32), we obtain

\[
\frac{d\tilde{c}}{dt} = -\left(\Gamma_{32} + k_2\right)\tilde{c} + \frac{1}{\Gamma_{32} + k_2} \left[k_1\hat{e}_2(t) - k_2\hat{e}_3(t) + k_3\hat{c}(t)\right]
\]

because \(c(t)\) in (30) is bounded. Since \(\Gamma_{21}, \Gamma_{32}, k_1, k_2,\) and \(k_3\) are positive constants and the observer gains \(L_1\) and \(L_2\) are chosen such that 
\[
\begin{pmatrix}
-\Gamma_{32} & L_1B_i \\
\Gamma_{32} & -\Gamma_{32} - L_2B_i
\end{pmatrix}
\]

is stable, the error equations given by (88)-(90) and (93) satisfy the BIBO stability. Therefore, the error state vector \(\hat{E}\) and \(\hat{E}_3\) defined by (85) are bounded and there exist positive constants \(B_E\) and \(B_{E3}\) such that

\[
\|\hat{E}(t)\| \leq B_E \|\hat{E}_3(t)\| \leq B_{E3}, \forall t \geq 0.
\]

As in step 2, we now consider the performance for \(t \in [t_i, t_{i+1}), i = 1, \ldots, M\). Since each channel input \(p_{in}^k\), \(k = 1, \ldots, N\) is zero or positive constant for all \(t \in [t_i + \epsilon, t_i], i = 1, \ldots, M\), the time derivative of \(c(t)\) is given by

\[
\begin{align*}
\frac{d}{dt}c(t) &= \sum_{k=1}^{N} (-e_{G(t)}^k B_k p_{1n}^k) (\hat{e}_2(t)) \\
&= \Pi_i \left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32} + k_2}\right) (\hat{E}_2 + \hat{e}_2) \\
&\quad - \Pi_i \frac{\Gamma_{32}k_2}{\Gamma_{32} + k_2} (\hat{E}_3 + \hat{e}_3) - \Pi_i \Gamma_{32}\hat{E}_3 - \Pi_i \hat{E}_C \\
\end{align*}
\]

where

\[
\Pi_i(t) = \sum_{k=1}^{N} B_ke_{G_i(t)} p_{1n}^k.
\]

As in step 2, the following error equation holds for all \(t \in [t_i + \epsilon, t_i], i = 1, \ldots, M\):

\[
\begin{align*}
\frac{d\hat{E}_C}{dt} &= -A_{D}\hat{E}_C + \hat{c}. \\
&= -A_{D}\hat{E}_C + \Pi_i \left(\Gamma_{21} + \frac{\Gamma_{32}k_1}{\Gamma_{32} + k_2}\right) (\hat{E}_2 + \hat{e}_2) \\
&\quad - \Pi_i \frac{\Gamma_{32}k_2}{\Gamma_{32} + k_2} (\hat{E}_3 + \hat{e}_3) - \Pi_i \Gamma_{32}\hat{E}_3 - \Pi_i \hat{E}_C \\
\end{align*}
\]

Define a Lyapunov-like function \(V_2\) by

\[
V_2(t) = X_2^T P_2 X_2
\]

where

\[
X_2 = \begin{bmatrix} \hat{E}^T & \hat{E}_3 \end{bmatrix}^T, P_2 = \text{diag}(\rho_3, \rho_3), \rho_3 > 0.
\]

Then, it follows from (36), (49), (69), (71), (73), and (74) that for all \(t \in [t_i + \epsilon, t_i], i = 1, \ldots, M\),
\[
\frac{d}{dt} V_2(t) \leq -\tilde{\mathbf{E}}^T \bar{Q}(t) \tilde{\mathbf{E}} + 2\tilde{\mathbf{E}}^T P_3 \mathbf{B}_3 \tilde{\mathbf{E}}_3 + 2\tilde{\mathbf{E}}^T P_3 \mathbf{B}_3 \mathbf{E}_{ee} \\
-2\rho_3 \left( \Gamma_{32} + k_2 - \frac{\Gamma_{32} k_1}{\Gamma_{32} + k_2} \right) \tilde{\mathbf{E}}_3^2 + \frac{2k_2}{\Gamma_{32} + k_2} \rho_3 L_4 \mathbf{B}_4 \tilde{\mathbf{E}}_3 \\
-2\rho_3 \left( \frac{k_1}{\Gamma_{32} + k_2} \right) \left( \Gamma_{21} + \frac{\Gamma_{32} k_1}{\Gamma_{32} + k_2} \right) \tilde{\mathbf{E}}_3 \left( \mathbf{E}_2 + \mathbf{E}_2 + \tilde{\mathbf{E}}_5 \right) \\
+2\rho_3 \left( \frac{\Gamma_{32} k_2}{\Gamma_{32} + k_2} \right) \mathbf{E}_5 \tilde{\mathbf{E}}_3 \\
+2\rho_3 \mathbf{P}_1 \left( \Gamma_{21} + \frac{\Gamma_{32} k_1}{\Gamma_{32} + k_2} \right) \tilde{\mathbf{E}}_5 \left( \mathbf{E}_2 + \mathbf{E}_2 + \tilde{\mathbf{E}}_5 \right) \\
+2\rho_3 \mathbf{P}_1 \mathbf{P}_3 \left( \frac{\Gamma_{32} k_2}{\Gamma_{32} + k_2} \right) \mathbf{E}_5 \tilde{\mathbf{E}}_3
\]

where

\[
\mathbf{B}_3 = \mathbf{B}_{31} + \mathbf{B}_{32} \left( \frac{k_2}{\Gamma_{32} + k_2} \right) + \mathbf{B}_{33} \Pi_1 \\
\mathbf{B}_{31} = \left[ \delta_1 \ 0 \ 0 \ \delta_4 \right]^T, \quad \mathbf{B}_{32} = \left[ 0 \ \delta_3 \ \delta_2 \ 0 \right]^T, \\
\mathbf{B}_{33} = \left[ 0 \ 0 \ 0 \ -\Gamma_{32} \right]^T, \quad \bar{\mathbf{B}}_2 = \left[ -1 \ 0 \ -1 \ 0 \right]^T, \\
\delta_1 = -\frac{\lambda (\lambda - \Gamma_{31})}{\Gamma_{32}}, \quad \delta_2 = -L_4 B_4, \\
\delta_3 = \lambda - \Gamma_{21} + \Gamma_{32}, \quad \delta_4 = \left( \frac{\lambda - \Gamma_{31}}{\Gamma_{32}} \right) A_0.
\]

By (79), for all \( t \in [t_i, t_{i+1}), i = 1, \cdots, M \),

\[
\frac{d}{dt} V_2(t) \leq -\tilde{\mathbf{E}}^T \bar{Q}(t) \tilde{\mathbf{E}} + 2\tilde{\mathbf{E}}^T P_3 \mathbf{B}_3 \tilde{\mathbf{E}}_3 + 2\tilde{\mathbf{E}}^T P_3 \mathbf{B}_3 \mathbf{E}_{ee} \\
-2\rho_3 \left( \lambda + \Gamma_{21} + \Gamma_{32} + k_2 \right) \tilde{\mathbf{E}}_3^2 + \frac{2k_2}{\Gamma_{32} + k_2} \rho_3 L_4 \mathbf{B}_4 \tilde{\mathbf{E}}_3 \\
+2\rho_3 \left( \frac{k_1}{\Gamma_{32} + k_2} \right) \left( \Gamma_{21} + \frac{\Gamma_{32} k_1}{\Gamma_{32} + k_2} \right) \tilde{\mathbf{E}}_3 \left( \mathbf{E}_2 + \mathbf{E}_2 + \tilde{\mathbf{E}}_5 \right) \\
+2\rho_3 \left( \frac{\Gamma_{32} k_2}{\Gamma_{32} + k_2} \right) \mathbf{E}_5 \tilde{\mathbf{E}}_3 \\
+2\rho_3 \mathbf{P}_1 \left( \Gamma_{21} + \frac{\Gamma_{32} k_1}{\Gamma_{32} + k_2} \right) \tilde{\mathbf{E}}_5 \left( \mathbf{E}_2 + \mathbf{E}_2 + \tilde{\mathbf{E}}_5 \right) \\
+2\rho_3 \mathbf{P}_1 \mathbf{P}_3 \left( \frac{\Gamma_{32} k_2}{\Gamma_{32} + k_2} \right) \mathbf{E}_5 \tilde{\mathbf{E}}_3
\]

Let \( 0 < \lambda < \bar{\lambda} \). In order for \( \tilde{\mathbf{Q}}(t) - \bar{\lambda} I_{4\times 4} \) to be positive definite for all \( t \geq 0 \), the following inequality must be satisfied:

\[
2\rho_3 \left( k_2 - \lambda - \Gamma_{21} + \Gamma_{32} - \bar{\lambda} \right) (\bar{\lambda} - \lambda) \\
- \left( B_{31} + \frac{k_2}{\Gamma_{32} + k_2} + B_{33} \Pi_1 \right)^T P_3 P_1 \left( B_{31} + \frac{k_2}{\Gamma_{32} + k_2} + B_{33} \Pi_1 \right) > 0
\]

Define \( \sigma_M \) by

\[
\sigma_M = \max_{m_{1}} \left( B_{31} + \frac{k_2}{\Gamma_{32} + k_2} + B_{33} \Pi_1 \right)^T P_3 P_1 \left( B_{31} + \frac{k_2}{\Gamma_{32} + k_2} + B_{33} \Pi_1 \right)
\]

Since \( \Pi_1(t) \) is positive and bounded as in (67), \( \sigma_M \) is obtained for each \( k_2 \) when \( \Pi_1(t) = \Pi_m \) or \( \Pi_1(t) = \Pi_M \). Let us define \( \sigma_1 \) and \( \sigma_2 \) as

\[
\sigma_1(k_2) = \left( B_{31} + \frac{k_2}{\Gamma_{32} + k_2} + B_{33} \Pi_m \right)^T P_3 P_1 \left( B_{31} + \frac{k_2}{\Gamma_{32} + k_2} + B_{33} \Pi_m \right)
\]

\[
\sigma_2(k_2) = \left( B_{31} + \frac{k_2}{\Gamma_{32} + k_2} + B_{33} \Pi_M \right)^T P_3 P_1 \left( B_{31} + \frac{k_2}{\Gamma_{32} + k_2} + B_{33} \Pi_M \right)
\]

If \( k_2 \) is chosen such that

\[
2\rho_3 \left( k_2 - \lambda + \Gamma_{21} + \Gamma_{32} - \bar{\lambda} \right) (\bar{\lambda} - \lambda) - \sigma_1(k_2) > 0
\]
and

\[ 2\rho_3 (k_2 - \lambda + \Gamma_{21} + \Gamma_{32} - \lambda) \left( \hat{x} - \lambda \right) - \sigma_2 (k_2) > 0, \]  

(108)

then \((\hat{Q}(t) - \hat{\lambda} I_{5 \times 5})\) is positive definite. Since the inequalities (107) and (108) are of third order in \(k_2\) if \((\Gamma_{32} + k_2)^2\) is multiplied to both sides of the inequalities, there always exists a \(k_2\) satisfying (107) and (108). Then, for all \(t \in [t_i + \varepsilon, t_{i+1}), i = 1, \cdots, M\),

\[
\frac{d}{dt} V_2(t) \leq -\hat{\lambda} \hat{E}^T \hat{E} + 2\rho_3 \left( \frac{\Gamma_{33} k_2}{\Gamma_{32} + k_2} \right) \| \hat{E} \| + 2 \| \rho_3 B_2 \| \| \hat{E} \| \| \hat{E} \| \]  

\[
+ 2\rho_3 \left( \Gamma_{21} + \frac{\Gamma_{31} k_2}{\Gamma_{32} + k_2} \right) \| \hat{E} \| \left( \| E_1 \| + \| E_2 \| + \| E_2 \| \right) 
+ 2\rho_3 \Pi_1 \left( \Gamma_{31} + \frac{\Gamma_{33} k_2}{\Gamma_{32} + k_2} \right) \| \hat{E} \| \left( \| E_1 \| + \| E_2 \| + \| E_2 \| \right) 
+ 2\rho_3 \Pi_1 \left( \Gamma_{31} + \frac{\Gamma_{33} k_2}{\Gamma_{32} + k_2} \right) \| \hat{E} \| \left( \| E_1 \| + \| E_2 \| + \| E_2 \| \right) \]  

(109)

Using the same arguments as in step 2, it can be shown from (51), (55), (56), (71), (82), and (109) that there exist positive numbers \(\tilde{a}_{2,i}\) and \(\tilde{b}_{2,i}, i = 1, \cdots, M\) satisfying

\[
\| X_2(t) \| \leq \tilde{a}_{2,i} \| X_2(t_i + \varepsilon) \| e^{-\tilde{b}_{2,i}(t_i - t_i)}, \]  

\forall t \in [t_i + \varepsilon, t_{i+1}). \]  

(110)

From (94), there exists a positive constant \(B_{X2}\) such that

\[
\| X_2(t) \| \leq B_{X2}, \forall t \geq 0 \]  

(111)

Thus, asymptotic stability is achieved since (110) holds for \(t \in [t_M + \varepsilon, \infty)\). Hence,

\[
e_2 = r_2 - r_2^* = \hat{E}_2 + \tilde{E}_2 + E_2 \]  

(112)

and from (55), (83), and (110)

\[
e_2 \to 0, r_2 \to r_2^* \]  

(113)

as \(t \to \infty\). By (10) and (12),

\[
G_k(t) \to G_k^c \]  

(114)

as \(t \to \infty\). This completes the stability analysis of \(\Sigma_1\).

### 4. Simulations

In order to analyze the performance of the proposed method, computer simulations are carried out. In the simulations, the wavelength of the pump laser is 980nm. As for signals, two channel signals with 1552.4 nm and 1557.9 nm wavelengths are applied to the system. The signal power of each channel is 0.316mW. In the simulations, the desired channel 1 signal gain is set to 6.6897. The other EDFA system parameters in (8), (9), and (10) are given as follows.

\[
\Gamma_{21} = 95.2381, \Gamma_{32} = 9.5238 \times 10^{-5}, 
A_1 = 5.0750, B_1 = 6.4998 \times 10^{-16}, 
A_2 = 4.3750, B_2 = 5.9624 \times 10^{-16}, 
A_p = 8.9950, B_p = 4.6900 \times 10^{-16}. \]  

(115)

Since the gain control is desired to be achieved within a microsecond, the controller gains \(k_1\) and \(k_2\) are chosen as follows

\[
k_1 = 1.05 \times 10^8, k_2 = 1.9048 \times 10^7 \]  

(116)

so that the natural frequency of the resultant second-order closed loop system may become \(\omega_n = 10^7\) (rad / sec). Thus, \(k_3\) is set to be 20.9999 from (36). Next, observer gains \(L_1\) and \(L_2\) are designed such that the bandwidth of the observer system is almost three times larger than that of state feedback control system. Observer gains \(L_1\) and \(L_2\) are given by

\[
L_1 = 1.3630 \times 10^{24}, L_2 = 9.0845 \times 10^{22} \]  

(117)

Finally, the channel add/drop estimator gain \(A_{D}\) in (25) or (26) is selected to be \(A_{D} = 1.5 \times 10^8\) in order for the channel add/drop estimation to be fast enough compared with other controller and observer. In this case, \(A_{D} = 1.5 \times 10^8\) is chosen such that the bandwidth is 5 times larger than that in state observer.

Firstly, we show that the performance of the proposed controller based on the three-level model is superior to that of the controller designed based on a simplified two-level model. In order to show this, we consider the following simple error state feedback controller including the same channel add/drop estimator.

\[
\hat{P}_p(t) = \frac{1}{1 - e^{\rho_3(t)}} \left[ k_2 e_2(t) - \bar{c}(t) - \Gamma_{21} r_2^* \right] \]  

(118)

As in the selection of the gains of the proposed controller, the gain \(k_c\) in (114) is also chosen as large as possible so that the gain control of the resultant first order control system can be achieved within a microsecond. For
example,

$$k_c = 1.0 \times 10^8$$  \hspace{1cm} (119)

Fig. 2 shows the graphs of the controlled gain of channel 1 signal when channel add/drop occurs at every microsecond as in Fig. 4. As expected, the proposed observer based controller designed based on the three-level model shows faster settling performance. The control based on the two-level model shows oscillation and longer settling performance because it considers the model simplified by ignoring the level three state.

The proposed controller guarantees the desired performance with 0.8 usec settling time, but the simplified control cannot guarantee the desired settling performance with a settling time longer than 1 usec. Fig. 3 is an enlarged version of Fig. 2 to show the gain control results and the influence on the gain due to channel add/drops was effectively compensated for within 1 usec as expected. The channel add/drop estimation performance is shown in Fig. 4. The channel add/drop estimation should be fastest compared with gain control and state observation and Fig. 4 shows that the estimation is done abruptly. In more details, Fig. 5 and Fig. 6 show the channel add/drop estimation results in case of channel drop and channel add. In both cases, the channel add/drop estimation is achieved in 0.03 microsecond and the channel gain is stabilized within 1 microsecond. Fig. 7 shows the results of state estimations. As we intended, the state estimation is done in 0.15 microsecond which is 5 times larger than channel add/drop estimation.
5. Conclusion

In this paper, a systematic design methodology of an EDFA gain controller has been proposed based on singular perturbation and observer technique. The three-level EDFA model has been fully considered without any simplification, and time scaling design approach based on singular perturbation technique has been applied.

Theoretical stability analysis has been carried out thoroughly. Through computer simulation, it is shown that the performance of the proposed EDFA controller is superior to that of the controller designed based on the simplified two-level model. The computer simulation also shows that the well-known disturbance observer technique plays an effective role in guaranteeing the desired performance when channel add/drops occur.

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